the discontinuities do not agree with the characteristic velocities, then the number of characteristics leaving both discontinuities in the domain separating them will equal the order of the system, i.e., the number of independent variables characterizing the state between the discontinuities. If all quantities characterizing the state between the discontinuities are eliminated from the relationships on the discontinuities (at least mentally), then the number of remaining relationships connecting the quantities with the external sides of the system of discontinuities and the velocities W_1 and W_2 of the discontinuities will equal the number of characteristics departing to the outside plus two. Just as many relationships are evidently needed to find the perturbations outside the system of discontinuities and the perturbations of their velocities.

As already noted, if $W_1 = W_2$, then the sequence of such two discontinuities could be considered one discontinuity with all the conservation laws satisfied on it. This discontinuity is evidently non-evolutionary since, according to the above, the number of boundary conditions thereon exceeds the number of characteristics leaving it by two. Moreover, upon actual interaction with small perturbations, the velocities W_1 and W_2 can receive different increments, the jump is split and the perturbations cease to be small. If W_1 and W_2 are considered to be identically equal in the relations on the discontinuity (i.e., it is considered that the increments of these quantities are also equal), then the solution of the problem of interaction between the discontinuity and arbitrary small perturbations will not exist.

By reasoning similar to that presented above, it can be seen that if there are m evolutionary discontinuities moving at the same velocity $W_1 = W_2 = \ldots = W_m$, then the number of independent relationships on such a discontinuity from which quantities characterizing the state between the discontinuities are eliminated (or did not enter from the very beginning), should exceed by m the number of characteristics leaving such a combined discontinuity.

Hence, the following recommendation can be formulated. If there is a discontinuity on which the known relationships (following from the conservation laws, say) are too many for evolutionarity as a single discontinuity, then several evolutionary discontinuities should be sought which will turn into the discontinuity of interest to us when their velocities are equal.

REFERENCES

1. KULIKOVSKII A.G. and SVESHNIKOVA E.I., On shocks propagating in a state of stress in isotropic non-linearly elastic media, PMM, Vol.44, No.3, 1980.
2. KULIKOVSKII A.G. and SVESHNIKOVA E.I., Investigation of the shock adiabat of quasitransverse shocks in a prestressed elastic medium, PMM, Vol.46, No.5, 1982.
3. LAX P.D., Hyperbolic systems of conservation laws. II, Communs. Pure and Appl. Math., Vol.10, No.4, 1957.
4. AKHIEZER A.I., LUBARSKII G. YA. and POLOVIN R.V., On the stability of shocks in magnetohydrodynamics, Zh. Eksp. Teor. Fiz., Vol.35, No.3, 1958.

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THE CONTACT-HYDRODYNAMIC PROBLEM OF LUBRICATION THEORY FOR ELASTIC BODIES WITH CRACKS *

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A mechanical model of lubricating solid bodies weakened by cracks is proposed. The model can be used to explain the reason for fatigue-induced crumbling of the surfaces. The presence of boundary and subsurface cracks is taken into account, and the interaction of the lubricant with elastic bodies within the cavities of boundary cracks is regarded as the most interesting aspect of the problem. Conditions are obtained characterizing the actual behaviour of the lubricant within the crack cavities, taking into account the pressure rise in the closed cavities completely filled with the lubricant and the possible onset of cavitation. The problem is reduced to a system of non-linear integro-differential and linear integral equations with additional conditions in the form of equations and inequalities.

The method of regular perturbations is used to study the state of weakly loaded elastohydrodynamic contact. In this case the problem is reduced to a sequence of purely hydrodynamic boundary value problems for the non-linear or linear ordinary differential equations, and elastic problems for the linear integral equations with one-sided constraints.

The effect of the temperature and lubricant on the contact stresses, taking the roughness of the bodies into account, was analysed in /1-3/ and the development of cracks and their influence on long-term fatigue in /4-7/.

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1. Formulation of the problem.

Consider the plane isothermal problem of two infinite circular cylinders rolling slowly on each other. The cylinders have parallel generatrices, with radii R_1 and R_2, and are separated by a thin layer of lubricant (an infinite cylindrical cavity in an elastic body can serve as one of the cylinders). A compressive linear force P acts on the cylinders. We shall assume the cylinders to be smooth and made of the same elastic material weakened by the cracks. We assume for simplicity that the lubricant is an incompressible Newtonian fluid.

The above assumptions together with the assumption that the lubricant layer is thin compared with its spread /1,3/, yield the following Reynolds equation:

$$\frac{d}{dx}\left(\frac{h^3}{12\mu}\frac{dp}{dx}\right)=\frac{u_1{}^\circ+u_2{}^\circ}{2}\frac{dh}{dx} \tag{1.1}$$

describing the behaviour of the lubricant in the contact area. When deriving (1.1), we also assumed that the cavities formed by the cracks open to the surface (boundary cracks) are filled with lubricant. In (1.1) x is the abscissa in a coordinate system attached to the median line in the lubricant layer whose ordinate passes through the centres of curvature of the cylinders $p=p(x)$ is the contact pressure, $h=h(x)$ is the gap between the bodies in contact, $\mu=\mu(p)$ is the lubricant dynamic viscosity coefficients, and $u_1{}^\circ$ and $u_2{}^\circ$ are the linear velocities of the points on the surfaces of the lower and upper body. Figure 1 depicts the rolling cylinders and pressure between them.

We shall consider the stress-strain state in elastic bodies, assuming for simplicity that there are no cracks in the upper cylinder, and N rectilinear cracks in the lower cylinder. We also assume that 1) there is no friction at the crack edges of 2) there is no friction and the edges of open cracks and full coupling occurs at the closed segments of the crack edges. The first type of boundary conditions corresponds to the weak, and second to the strong interaction of the material of the bodies.

Taking into account the pressure $p(x)$ applied to the boundaries of the bodies in contact we find, that under the above assumptions, the stress-strain state of the cracks in the lower body is described, in the quasistationary approximation, by the system of equations /6,7/

$$\int_{-l_n}^{l_n}\frac{v_n'(t)\,dt}{t-x_n}+\sum_{k=1}^{N}\int_{-l_k}^{l_k}\{v_k'(t)\,U_{nk}^r(t,x_n)-u_k'(t)\,V_{nk}^r(t,x_n)\}\,dt=\frac{4\pi}{E'}\,p_n(x_n)-\frac{4}{E'}\int_{x_i}^{x_e}p(\tau)\,D_n{}^r(\tau,x_n)\,d\tau \tag{1.2}$$

$$\int_{-l_n}^{l_n}\frac{u_n'(t)\,dt}{t-x_n}+\sum_{k=1}^{N}\int_{-l_k}^{l_n}\{-u_k'(t)\,V_{nk}^i(t,x_n)+v_k'(t)\,U_{nk}^i(t,x_n)\}\,dt=-\frac{4}{E'}\int_{x_i}^{x_e}p(\tau)\,D_n{}^i(\tau,x_n)\,d\tau;\quad n=1,2,\ldots,N$$

$$U_{nk}^r+iU_{nk}^i=\overline{R_{nk}+S_{nk}},\quad V_{nk}^r+iV_{nk}^i=-\overline{i(S_{nk}-R_{nk})} \tag{1.3}$$

$$D_n{}^r+iD_n{}^i=\overline{D}_n$$

$$R_{nk}(t,x_n)=(1-\delta_{nk})\,K_{nk}(t,x_n)+\frac{e^{i\alpha_k}}{2}\left\{\frac{1}{\overline{X}_n-\overline{T}_k}+\frac{e^{-2i\alpha_n}}{X_n-\overline{T}_k}+\right.$$
$$\left.(\overline{T}_k-T_k)\left[\frac{1+e^{-2i\alpha_n}}{(\overline{X}_n-\overline{T}_k)^2}-\frac{2e^{-2i\alpha_n}(X_n-T_k)}{(\overline{X}_n-\overline{T}_k)^3}\right]\right\}$$

$$S_{nk}(t,x_n)=(1-\delta_{nk})\,L_{nk}(t,x_n)+\frac{e^{-i\alpha_k}}{2}\left[\frac{T_k-\overline{T}_k}{(\overline{X}_n-\overline{T}_k)^2}+\frac{1}{\overline{X}_n-\overline{T}_k}-e^{-2i\alpha_n}\frac{X_n-T_k}{(\overline{X}_n-\overline{T}_k)^2}\right]$$

$$K_{nk}(t,x_n)=\frac{e^{i\alpha_k}}{2}\left(\frac{1}{T_k-X_n}+\frac{e^{-2i\alpha_n}}{\overline{T}_k-\overline{X}_n}\right)$$

$$L_{nk}(t,x_n)=\frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k-\overline{X}_n}-\frac{T_k-X_n}{(\overline{T}_k-\overline{X}_n)^2}e^{-2i\alpha_n}\right]$$

$$D_n(\tau,x_n)=\frac{i}{2}\left[-\frac{1}{\tau-X_n}+\frac{1}{\tau-\overline{X}_n}-\frac{e^{-2i\alpha_n}(X_n-\overline{X}_n)}{(\tau-\overline{X}_n)^2}\right]$$

$$X_n=x_ne^{i\alpha_n}+z_n{}^\circ,\quad T_k=te^{i\alpha_k}+z_k{}^\circ \tag{1.4}$$

$$z_k{}^\circ=x_k{}^\circ+i\left[y_k{}^\circ+\frac{1}{2}h(x_k{}^\circ)\right]$$

Figure 2 shows the general pattern of crack distribution in an elastic body, and the relative position of the fundameneal and local coordinate system. Here x_n is the abscissa of the local coordinate system associated with the n-th crack, and α_n is the angle between the abscissas of the local and fundamental coordinate systems, $z_n{}^\circ$ are the complex coordinates of the origin of the local reference system associated with the n-th crack, l_n is the half-length of the n-th crack, v_n and u_n are the corresponding jumps in the normal and tangential displacement of the edges of the n-th crack, p_n is normal force applied to the edges of the n-th crack, $E'=E/(1-\nu^2)$ is the normalized modulus of elasticity of the body material, x_i and x_e are the coordinates of the points of entry to and exit from the region of contact in the fundamental coordinate system, and δ_{nk} is the Kronecker delta.

Let us now analyse the conditions which must be added to (1.2)-(1.4), depending on whether the cracks are boundary or subsurface, and whether their edges close or not.

We shall consider subsurface cracks first. At the open segments of the cracks the normal stresses are zero, and positive at the closed segments. As a result we obtain the following system of alternating equations and inequalities:

$$\begin{aligned} &p_n(x_n)=0, \quad v_n(x_n)>0, \\ &p_n(x_n)\leqslant 0, \quad v_n(x_n)=0, \end{aligned} \quad |l_n \sin\alpha_n| < \left| y_n^\circ + \frac{1}{2} h(x_n^\circ) \right| \tag{1.5}$$

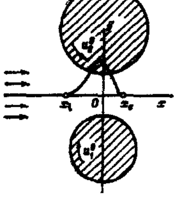

Fig.1

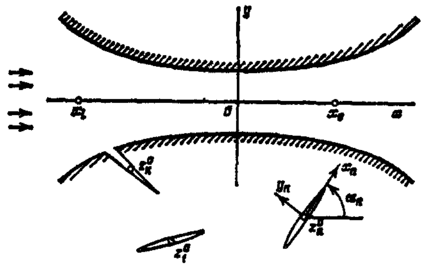

Fig. 2

In the case of type 2) boundary conditions we must supplemenr relations (1.5) at the closed segments of the cracks, with the condition

$$u_n(x_n) = 0,\ v_n(x_n) = 0;\ |l_n \sin\alpha_n| < |y_n^\circ + {}^1/_2 h(x_n^\circ)| \tag{1.6}$$

Let us pause and consider the case of boundary cracks. Let us consider the n-th boundary crack whose tip emerges at the surface of the body at the point $x^\circ = x_n^\circ + l_n \cos\alpha_n \operatorname{sign}\alpha_n$. We shall say that the neck of the crack is open when $v_n(l_n \operatorname{sign}\alpha_n) > 0$ and closed when $v_n(l_n \operatorname{sign}\alpha_n) = 0$. We assume that the crack in question has several segments in which the edges are closed. The crack has a number of cavities containing the lubricant. The cavities are not connected to each other, nor to the lubricant layer covering the body. We shall assume that the lubricant within the cavities is in a state of hydrostatic equilibrium, and we require to establish the additional conditions at every segment of the crack mentioned above.

Physical considerations imply that $v_n(x_n) \geqslant 0$. Let us find the singly connected, non-intersecting sets of points $\operatorname{supp} v_n^i(x_n^\circ)$ for which $v_n(x_n) > 0$. We shall number these sets (cavities) beginning from the surface of the body. We have, in each of the sets $\operatorname{supp} v_n^i$, an inherent constant lubricant pressure, generally speaking not known in advance. We shall therefore denote the corresponding stresses at the crack edges caused by the lubricant pressure, by p_n^i $(i = 1, 2, \ldots)$. Let the neck of the crack be open, and the adjacent set be $\operatorname{supp} v_n^1$. Here the cavity $\operatorname{supp} v_n^1$ is in contact with the surface layer of the lubricant, therefore it is natural to assume that the pressure within the cavity $\operatorname{supp} v_n^1$ is equal to that within the neck. We have

$$\begin{aligned} &p_n^1 = -p(x_n^\circ + l_n \cos\alpha_n \operatorname{sign}\alpha_n), \quad v_n(l_n \operatorname{sign}\alpha_n) > 0 \\ &|l_n \operatorname{sign}\alpha_n| = |y_n^\circ + {}^1/_2 h(x_n^\circ)| \end{aligned} \tag{1.7}$$

Next we turn our attention to the cavities $\operatorname{supp} v_n^i$ which are not in contact with the lubricant surface layer through the neck. Clearly, the cavities not in contact with the lubricant layer are defined by the set of indices

$$I_n(x_n^\circ) = \left\{ i, \operatorname{supp} v_n^i \cap \left\{ (x, y), y = -\frac{h(x_n^\circ)}{2} \right\} = \varnothing,\ i = 1, 2, \ldots \right\} \tag{1.8}$$

We assume that the limiting tensile strength of the lubricant fluid is zero. Then the stresses in the crack cavities $p_n^i \leqslant 0$, $i \in I_n(x_n^\circ)$.

Consider the i-th cavity: $\operatorname{supp} v_n^i$, $i \in I_n(x_n^\circ)$. We note that when the body moves, its stress-strain state changes, so that the configuration of the i-th cavity of the n-th boundary crack also changes. However, the volume of the i-th cavity cannot be smaller than the volume of the lubricant filling the cavity. This follows from the incompressibility of the lubricant. Thus we have $V_n^i \geqslant V_{n0}^i$ where V_{n0}^i is the volume of the lubricant in the i-th cavity and

$$V_n^i = V_n^i(x_n^\circ) = \int_{\operatorname{supp} v_n^i} v_n(x_n)\, dx_n$$

is the volume of the i-th cavity. A more detailed study leads to following conclusions: if voids, i.e., lubricant-free volumes, appear in the i-th cavity, then $p_n^i=0$, while when the lubricant occupies the whole cavity, we have $p_n^i \leqslant 0$ (the vapour pressure of the lubricant fluid in the void and the surface tension of the lubricant are both neglected). We see the appearance of the cavitation phenomena, and as a result we have the system of alternate equations and inequalities

$$\begin{aligned} &p_n^i = 0, \quad V_n^i > V_{n0}^i, \\ &p_n^i \leqslant 0, \quad V_n^i = V_{n0}^i, \end{aligned} \quad i \in I_n(x_n^\circ);\ \ |l_n \sin\alpha_n| = |y_n^\circ + {}^1/_2 h(x_n^\circ)| \tag{1.9}$$

The method of determining the lubricant volume V_{n0}^i will be given below.

We note that conditions (1.9) for the i-th cavity hold, as long as the neighbouring cavities do not come into contact with it, i.e.

$$\operatorname{supp} v_n^i \cap \operatorname{supp} v_n^j = \varnothing,\ i \neq j;\ i, j \in I_n(x_n^\circ) \tag{1.10}$$

Relations (1.8)-(1.10) represent the necessary additional conditions, from which we obtain the a priori unknown stress p_n^i acting at the boundary of the cavity $\operatorname{supp} v_n^i$, $i \in I_n(x_n^\circ)$.

Next we consider the method of determining the volumes V_{n0}^i of the lubricant within the cavities $\operatorname{supp} v_n^i$. Previously, when deriving relations (1.9), we had in fact assumed that initially, i.e. when $\max x_n^\circ = -\infty$, all boundary cracks were open and filled with lubricant. As the bodies move, the cracks approach the region of contact $[x_l, x_r]$, at the same time changing their configuration.

Let us consider the behaviour of the n-th boundary crack during its motion. Let the edges of the n-th open crack be not in contact at the points $x = x_n^\circ - \varepsilon$ $(\varepsilon \to +0)$ where its centre is situated, and let the edges close at the point $x = x_n^\circ$ so that k cavities $\operatorname{supp} v_n^i$, $i = 1, 2, \ldots, k \in I_n(x_n^\circ)$ are formed simultaneously. Then the volumes $V_n^i(x_n^\circ)$ of these cavities will coincide with the volumes V_{n0}^i of the lubricant enclosed within them, i.e.

$$\begin{gathered} V_{n0}^i = \int\limits_{\operatorname{supp} v_n^i} v_n(x_n)\,dx_n, \quad p_n^i = -\lim_{\varepsilon\to+0} p(x_n^\circ - \varepsilon + l_n \cos\alpha_n \operatorname{sign}\alpha_n) \\ i = 1, 2, \ldots, k \in I_n(x_n^\circ);\ I_n(x_n^\circ - \varepsilon) = \varnothing,\ \varepsilon \to +0 \end{gathered} \tag{1.11}$$

The additional condition imposed on p_n^i is obtained from the assumption that the pressure in the lubricant fluid varies continuously.

Further, when k cavities $\operatorname{supp} v_n^i$, $i = i_0 + 1, \ldots, i_0 + k \in I_n(x_n^\circ - \varepsilon)$ merge simultaneously into a single cavity $\operatorname{supp} v_n^j$, $j \in I_n(x_n^\circ)$, as $\varepsilon \to +0$, we obtain

$$\begin{aligned} & V_{n0}^j = \sum_{i=i_0+1}^{i_0+k} V_{n0}^i, \quad j \in I_n(x_n^\circ); \\ & \operatorname{supp} v_n^l(x_n^\circ - \varepsilon) \cap \operatorname{supp} v_n^m(x_n^\circ - \varepsilon) = \varnothing \\ & l \neq m, \quad l, m \in I_n(x_n^\circ - \varepsilon), \quad \varepsilon > 0; \\ & \operatorname{supp} v_n^j(x_n^\circ) = \lim_{\varepsilon\to+0} \bigcup_{i=i_0+1}^{i_0+k} \operatorname{supp} v_n^i(x_n^\circ - \varepsilon) \end{aligned} \tag{1.12}$$

The case when the cavity $\operatorname{supp} v_n^j$ gives rise, at some point where the centre of the n-th boundary crack lies, simultaneously to k cavities $\operatorname{supp} v_n^i$, $i = i_0 + 1, \ldots, i_0 + k \in I_n(x_n^\circ)$, is more complicated. If for an arbitrarily small $\varepsilon > 0$ there were no voids $(V_n^j = V_{n0}^j)$ in the initial cavity, then by virtue of the continuity the cavities formed will also have no voids. Similarly, from continuity considerations we find the additional conditions for p_n^i. Thus we have

$$\begin{aligned} & V_{n0}^i = \int\limits_{\operatorname{supp} v_n^i} v_n(x_n)\,dx_n, \quad p_n^i = \lim_{\varepsilon\to+0} p_n^j(x_n^\circ - \varepsilon) \\ & i = i_0 + 1, \ldots, i_0 + k \in I_n(x_n^\circ); \\ & \bigcup_{i=i_0+1}^{i_0+k} \operatorname{supp} v_n^i(x_n^\circ) = \lim_{\varepsilon\to+0} \operatorname{supp} v_n^j(x_n^\circ - \varepsilon) \\ & V_n^j(x_n^\circ - \varepsilon) = V_{n0}^j, \quad j \in I_n(x_n^\circ - \varepsilon), \quad \varepsilon > 0 \end{aligned} \tag{1.13}$$

Let us now consider the case when the initial cavity contains a void, i.e. $V_n^j > V_{n0}^j$. We assume that in the presence of a void the lubricant layers are adsorbed on the boundaries of the cavity in volumes proportional to the volumes of the cavities. Keeping this in mind, we obtain

$$\begin{aligned} & V_{n0}^i = v \int\limits_{\operatorname{supp} v_n^i} v_n\,dx_n, \quad p_n^i = 0; \quad i_0 + 1, \ldots, i_0 + k \in I_n(x_n^\circ); \\ & \bigcup_{i=i_0+1}^{i_0+k} \operatorname{supp} v_n^i(x_n^\circ) = \lim_{\varepsilon\to+0} \operatorname{supp} v_n^j(x_n^\circ - \varepsilon); \quad V_n^j(x_n^\circ - \varepsilon) > V_{n0}^j \\ & j \in I_n(x_n^\circ - \varepsilon), \quad \varepsilon > 0; \quad v = V_{n0}^j \Big/ \int\limits_{\operatorname{supp} v_n^j} v_n\,dx_n \end{aligned} \tag{1.14}$$

Thus we have formulated all the necessary conditions within the cavities of the boundary cracks containing the lubricant.

When considering the segments of the boundary cracks with closed edges, we shall require that the following relations hold (the relations follow from the fact that no external forces act on the segments of the crack edges in question):

$$p_n(x_n) \leqslant 0,\ v_n(x_n) = 0,\ |l_n \sin\alpha_n| = |y_n^\circ + {}^1\!/_2 h(x_n^\circ)| \tag{1.15}$$

In addition, in the case of type 2) boundary conditions the following relations must hold:

$$u_n(x_n) = 0, \quad v_n(x_n) = 0, \quad |l_n \sin\alpha_n| = |y_n^\circ + {}^1\!/_2 h(x_n^\circ)| \tag{1.16}$$

Let us write the equation describing the gap between two contacting surfaces. Using the expression for the displacements of the half-plane boundaries with and without cracks, we find /1,7/

$$h(x) = c_* + \frac{4}{\pi E'}\int_{x_i}^{x_e} p(t)\ln\frac{1}{|x-t|}\,dt + \frac{1}{2\pi}\sum_{k=1}^{N}\int_{-l_k}^{l_k}\{-v_k'(t)W_k^r(t,x) + \tag{1.17}$$

$$u_k'(t)W_k^i(t,x)\}\,dt + \frac{x^2}{R'} \quad \left(\frac{1}{R'} = \frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right)$$

$$W_k^r = \operatorname{Re} W_k, \quad W_k^i = \operatorname{Im} W_k, \quad W_k(t,x) = ie^{-i\alpha_k}\frac{T_k - \bar{T}_k}{\bar{T}_k - x} \tag{1.18}$$

where c_* is an arbitrary constant, W_k^r, W_k^i and W_k are kernels and R' is the normalized radius of the contacting bodies.

The value of the constant c_* is chosen so that the following relation holds at the point of exit from the area of contact: $h(x_e) = h_e$ where h_e denotes the previously unknown lubricant layer thickness at the point $x = x_e$.

To close the equations of the problem we must add to them the conditions of contact pressure at the entry and exit points, and the condition of statics /1/

$$p(x_i) = p(x_e) = \frac{dp}{dx}(x_e) = 0, \quad \int_{x_i}^{x_e} p(x)\,dx = P \tag{1.19}$$

We note that the classical formulation of the contact-hydrodynamic problem in which the entry coordinate x_i is assumed known, is adopted here.

Thus with the constants $u_1^\circ, u_2^\circ, R', E', P, x_i, \{\alpha_k, x_k^\circ, y_k^\circ, l_k\}, k = 1, 2, \ldots, N$ and functions $\mu(p)$ given, we have to find from the solution of the problem the constants x_e, h_e and functions $p(x), h(x), \{v_k(x_k), u_k(x_k), p_k(x_k)\}, k = 1, 2, \ldots, N$.

Having solved the problem in terms of the known intensity coefficients $k_{ni}^\pm$ $(n = 1, 2, \ldots, N$ and $i = 1, 2)$, we can find the angles $\theta_n^\pm$ of the initial distribution of the cracks /5-7/.

Analysing the formulation of the problem given above we find, that the fatigue fracture of the material of the bodies can have two causes: 1) development of subsurface cracks and their emergence to the surface and 2) interaction of the lubricant with the boundary cracks leading to disintegration of the material. The idea of the development of subsurface cracks with consequent fatigue fracture was developed in many theoretical and experimental investigations (see /4,5/). It was assumed in the literature that the lubricant penetrating the boundary cracks acted as a wedge leading to fracture of the material. In the formulation used above the wedge effect of the lubricant appears when high pressure p_n^i is created within the cavity completely filled with lubricant $(V_n^i = V_{n0}^i)$, adjacent to the inner tip of the boundary crack. The excess pressure leads to development of the crack and subsequent fracture of the material. In a number of cases the process may lead to the well-known mechanism of fracture by peeling. However, another cavitation mechanism of fracture in contacting bodies is also possible. Indeed, if there are voids in the cavities of the boundary cracks containing the lubricant, then as we know /8/, considerable stresses appear in the lubricant near them when they collapse. These stresses also lead to fracture of the solid surface adjacent to the voids /8/. It appears in view of this, that cavitation in a number of cases, can play a major role in the fracture of lubricated surfaces.

An attempt was made earlier /9/ to formulate and solve a plane, contact-hydrodynamic problem of the theory of lubrication for the case when one of the contacting bodies has a boundary crack. However the additional conditions of the type (1.7)-(1.16) allowing for the behaviour of the lubricant within the cavities of the boundary crack were not formulated in /9/, and the investigation was actually carried out under the assumption that there were no such cavities. It is clear that this assumption does not always agree with reality and can therefore distort the picture of the phenomenon appreciably.

Thus we have formulated a plane, contact-hydrodynamic problem of lubrication theory, taking into account the mechanical effects arising within the boundary cracks of the elastic, lubricated body. The possibility of partial overlap of the cracks edges was also recognised and the problem was reduced to a non-linear boundary value problem with one-sided constraints.

2. Weakly loaded contact.

Let us consider the simplest case of a weakly loaded contact. We shall assume that the lubricated contact is weakly loaded if the influence of deformability of the elastic cylinders on the thickness of the lubricant layer and the contact pressure is small. The problem is studied using the method of regular asymptotic expansions /2,3,10/. Below we shall show that in the case of a weakly loaded contact the problems for the principal terms of the asymptotic expansions $p(x), h(x), x_e, h_e$ and principal terms of the asymptotic expansions $\{v_k(x_k), u_k(x_k), p_k(x_k)\}, k = 1, 2, \ldots, N$ are separable and can be solved one after the other. We shall also obtain and study the equations for further terms of the asymptotic expansions of $p(x), h(x), x_e$ and h_e.

Using the dimensionless variables /3/

$$\{x', \tau', a, c, x_k^{\circ\prime}, y_k^{\circ\prime}\} = \{x, \tau, x_i, x_e, x_k^\circ, y_k^\circ\}\frac{\theta}{R'}, \quad \{p', p_n', p_n^{i\prime}\} =$$

$$\{p, p_n, p_n^i\}\frac{\pi R'}{2\theta P}, \quad h' = \frac{h}{h_e}, \quad \mu' = \frac{\mu}{\mu_0}; \quad \theta^2 = \frac{P}{3\pi\mu_0(u_1^\circ + u_2^\circ)}$$

$$\{x_k', t'\} = \frac{\{x_k, t\}}{l_k}, \quad \{v_k', u_k'\} = \frac{\{v_k, u_k\}}{v_0};$$
$$k = 1, 2, \ldots, N; \quad v_0 = \frac{8P}{\pi E'}$$

(θ is a dimensionless constant) we write the equations and inequalities for the problem in question in the form (the primes are omitted)

$$\frac{d}{dx}\left(\frac{h^3}{\mu}\frac{dp}{dx}\right) = \frac{1}{\gamma^2}\frac{dh}{dx}; \quad p(a) = p(c) = \frac{dp}{dx}(c) = 0, \quad \int_a^c p(x)\,dx = \frac{\pi}{2} \tag{2.1}$$

$$\gamma(h-1) = x^2 - c^2 + \frac{2}{\pi V}\int_a^c p(t)\ln\frac{c-t}{|x-t|}\,dt + \tag{2.2}$$
$$\frac{1}{\pi V}\sum_{k=1}^{N}\int_{-1}^{1}\{-v_k'(t)[W_k^r(t,x) - W_k^r(t,c)] + u_k'(t)[W_k^i(t,x) - W_k^i(t,c)]\}\,dt$$

$$\frac{1}{\delta_n}\int_{-1}^{1}\frac{v_n'(t)\,dt}{t-x_n} + \sum_{k=1}^{N}\int_{-1}^{1}\{v_k'(t)\,U_{nk}^r(t,x_n) - u_k'(t)\,V_{nk}^r(t,x_n)\}\,dt = \pi p_n(x_n) - \int_a^c p(\tau)\,D_n^r(\tau,x_n)\,d\tau \tag{2.3}$$

$$\frac{1}{\delta_n}\int_{-1}^{1}\frac{u_n'(t)\,dt}{t-x_n} + \sum_{k=1}^{N}\int_{-1}^{1}\{-u_k'(t)V_{nk}^i(t,x_n) + v_k'(t)\,U_{nk}^i(t,x_n)\}\,dt = -\int_a^c p(\tau)\,D_n^i(\tau,x_n)\,d\tau; \quad n = 1, 2, \ldots, N$$

If $\delta_n|\sin\alpha_n| < \left|y_n^\circ + \frac{\gamma}{2\theta}h(x_n^\circ)\right|$, then (2.4)

$$\begin{cases} p_n(x_n) = 0 \text{ when } v_n(x_n) > 0 \\ p_n(x_n) \leqslant 0 \text{ when } v_n(x_n) = 0 \end{cases}$$

If $\delta_n|\sin\alpha_n| = \left|y_n^\circ + \frac{\gamma}{2\theta}h(x_n^\circ)\right|$, then

$p_n(x_n) = -p(x_n^\circ + \delta_n\cos\alpha_n \operatorname{sign}\alpha_n)$

$x_n \in \operatorname{supp} v_n^1(x_n^\circ)$ when $v_n(\operatorname{sign}\alpha_n) > 0$

$$\begin{cases} p_n(x_n) = 0 \quad \text{when } V_n^i(x_n^\circ) > V_{n0}^i, & x_n \in \operatorname{supp} v_n^i(x_n^\circ), \\ p_n(x_n) = p_n^i \leqslant 0 \text{ when } V_n^i(x_n^\circ) = V_{n0}^i, & i \in I_n(x_n^\circ) \end{cases}$$

$p_n(x_n) \leqslant 0$ when $v_n(x_n) = 0$; $n = 1, 2, \ldots, N$

(the regions $\operatorname{supp} v_n^i(x_n^\circ)$ are numbered beginning at the opening of the crack).

When type 2) boundary conditions hold at the crack edges, we must supplement relations (2.4) by the conditions

$$u_n(x_n) = 0 \text{ when } v_n(x_n) = 0; \; n = 1, 2, \ldots, N \tag{2.5}$$

The kernels $W_k^r, W_k^i, U_{nk}^r, V_{nk}^r, U_{nk}^i, V_{nk}^i, D_n^r, D_n^i$ in (2.2) and (2.3) have the form (1.3), while the quantities X_n, T_k and z_k° take the form

$$X_n = \delta_n x_n e^{i\alpha_n} + z_n^\circ, \quad T_k = \delta_k t e^{i\alpha_k} + z_k^\circ, \tag{2.6}$$
$$z_k^\circ = x_k^\circ + i\left[y_k^\circ + \frac{\gamma}{2\theta}h(x_k^\circ)\right]$$

In (2.4) we have assumed that

$$V_n^i(x_n^\circ) = \int_{\operatorname{supp} v_n^i(x_n^\circ)} v_n(x_n)\,dx_n, \quad i \in I_n(x_n^\circ) \tag{2.7}$$

$$\gamma = \frac{h_e\theta^2}{R'}, \quad V = \frac{3\pi^2}{4P^2}\mu_0(u_1^\circ + u_2^\circ)E'R', \quad \delta_n = \frac{l_n\theta}{R'} \tag{2.8}$$

and the constants V_{n0}^i can be found from the relations (1.11)—(1.14).

Large values of the parameter V correspond to the state of weak loading. For this reason we shall assume that $V \gg 1$ and seek the solution of the problem (2.1)—(2.7), (1.3), (1.11) — (1.14) in terms of asymptotic series in non-negative integral powers of V^{-1}. Carrying out the substitution with the independent variable

$$x = \frac{c+a}{2} + \frac{c-a}{2}y$$

we shall assume that

$$\{p, p_n, p_n^i, h, v_n, u_n, \gamma, c\} = \sum_{k=0}^{\infty}\{p^k, p_{nk}, p_{nk}^i, h_k, v_{nk}, u_{nk}, \gamma_k, c_k\}V^{-k} \tag{2.9}$$

Let us consider the simplest case $\mu = 1$. Then substituting (2.9) into the equations and inequalities of the problems and equating the coefficients of like powers of V, we obtain a series of problems for the consecutive terms of the asymptotic expressions of the solution. We obtain for p^0, h_0, γ_0 and c_0 /3/

$$p^0(x_0) = \frac{q(x_0) - q(a)}{8b^3}, \quad x_0 = \frac{c_0+a}{2} + \frac{c_0-a}{2}y, \quad b^2 = \gamma_0 - c_0^2 \tag{2.10}$$

$$q(x) = \frac{b^2 - 3c_0^2}{b^3}\left[\operatorname{arctg}\frac{x}{b} + \frac{bx}{b^2 + x^2}\right] - \frac{2b\gamma_0 x}{(b^2 + x^2)^2}$$

The constants γ_0 and c_0 are found from the solution of the following set of equations:

$$q(c_0) - q(a) = 0 \tag{2.11}$$

$$\frac{b^2 - 3c_0^2}{b^3}\left[\frac{c_0}{b}\left(\operatorname{arctg}\frac{c_0}{b} - \operatorname{arctg}\frac{a}{b}\right) - \frac{a(c_0 - a)}{a^2 + b^2}\right] + \frac{a^2 - c_0^2}{a^2 + b^2} + \frac{2a\gamma_0(c_0 - a)}{(a^2 + b^2)^2} - 4\pi b^2 = 0$$

Integrating the equtions for subsequent terms of the asymptotic expressions for p^1, h_1, γ_1 and c_1, we obtain

$$p^1(x_0) = \frac{1}{b^2}\left\{\frac{3\gamma_1}{b^2}\varepsilon_\gamma(x_0) + \frac{c_1}{c_0 - a}\varepsilon_c(x_0) + \frac{\gamma_0}{b^2}\varepsilon(x_0)\right\} \tag{2.12}$$

where the constants γ_1 and c_1 are found from the solution of the following set of linear equations:

$$3\varepsilon_\gamma(c_0)\gamma_1 + \frac{b^2}{c_0 - a}\varepsilon_c(c_0)c_1 = -\gamma_0\varepsilon(c_0) \tag{2.13}$$

$$3\int_a^{c_0}\varepsilon_\gamma(x)\,dx\cdot\gamma_1 + \frac{b^2}{c_0 - a}\left[\int_a^{c_0}\varepsilon_c(x)\,dx + \frac{\pi b^2}{c_0 - a}\right]c_1 = -\gamma_0\int_a^{c_0}\varepsilon(x)\,dx$$

The expressions for the functions $\varepsilon_\gamma(x)$ and $\varepsilon_c(x)$ from (2.12) and (2.13) are as accurate as those given in /3/, and the function $\varepsilon(x)$ has the form

$$\varepsilon(x) = b^2\int_a^x \frac{Q(t)}{H^2(t)}\left[\frac{3\gamma_0}{H(t)} - 2\right]dt, \quad H(t) = \gamma_0(h_0(t) - 1) = b^2 + t^2 \tag{2.14}$$

$$Q(x) = \frac{2}{\pi}\int_a^{c_0} p^0(t)\ln\frac{c_0 - t}{|x - t|}\,dt + \frac{1}{\pi}\sum_{k=1}^{N}\int_{-1}^{1}\{-v_{k0}'(t)[W_{k0}^r(t,x) - W_{k0}^r(t,c_0)] + u_{k_0}'(t)[W_{k0}^i(t,x) - W_{k0}^i(t,c_0)]\}\,dt$$

Since the relations for determining the principal terms $p_{n0}, p_{n0}^{\,i}, v_{n0}$ and u_{n0} of the asymptotic expressions occurring in (2.4) are linear, it follows that they are identical with (2.3)–(2.7), (1.3), (1.11)–(1.14) provided that a subscript zero is added to all unknown quantities. We note that the right hand sides of the relations shown contain the quantities $p^0(x), h_0(x), \gamma_0$ and c_0 already determined from (2.10) (2.11) and the second relation of (2.14)

Thus we have succeeded, in the case of a weakly loaded contact, in reducing the initial problem for the bodies with cracks, to a sequence of purely hydrodynamic and purely elastic problems. When $\mu = 1$ the initial problem was reduced to that of solving system (2.11), then solving the problem of, the stress-strain state of an elastic half-plane with cracks (2.3) – (2.7), (1.3), (1.11) – (1.14) at $p = p^0, h = h_0, \gamma = \gamma_0$ and $c = c_0$, and finally system (2.13).

We stress that the principal terms $p^0(x), h_0(x), \gamma_0$ and c_0 of the asymptotic expansions are independent of the presence of cracks in the bodies, and the subsequent terms of the asymptotic expansions, e.g. $p^1(x), h_1(x), \gamma_1$ and c_1, depend on the distribution of the cracks relative to each other and to the region of contact $[a, c_0]$. For this reason we shall have, particularly in the quasistatic formulation adopted here, $\gamma_1 = \gamma_1(x_1^\circ)$ and $c_1 = c_1(x_1^\circ)$.

Clearly, when the body contains only subsurface cracks, we can come to a qualitative conclusion concerning the increase in the thickness of the lubricant layer h_e as compared with the case when the body has no cracks. At the same time, it is hardly possible to comment on the behaviour of the thickness of the lubricant layer h_e when the bodies contain boundary cracks.

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REFERENCES

1. KODNIR D.S., Contact Hydrodynamics of the Lubrication of Machine Parts. Moscow, Mashinostroenie, 1976.
2. KUDISH I.I., On solution of some contact and elastohydrodynamic problems. In: Tribological Process in Contact Areas of Lubricated Solid Bodies. 3-rd Intern. Tribology Congr. EUROTRIB 81. V. 2. W-wa: Wyd-wo kommunikacji i lacznosci, 1981.
3. KUDISH I.I., Certain problems of the elastohydrodynamic theory of the lubrication of a weakly loaded contact. Izv. Akad. Nauk SSSR, MTT, No.3, 1981.
4. LUNDBERG G. and PALMGREN A., Dynamic Capacity of rolling Bearings. Acta Polytech. Ser. Mech. Engng, No.7, 1947.
5. CHEREPANOV G.P., Mechanics of Brittle Fracture. Moscow, Nauka, 1974.
6. PANASYUK V.V., SAVRUK M.P. and DATSYSHIN A.P., Stress distribution near the cracks in plates and shells. Kiev, Naukova Dumka, 1976.
7. SAVRUK M.P., Two-dimensional problems of elasticity for bodies with cracks. Kiev, Naukova Dumka, 1981.
8. BATCHELOR G.K., An Introduction to Fluid Dyanamics. Cambridge University Press, Cambridge, 1967.
9. ROMANIS N.B., Solution of problems of crumbling surfaces in bodies in contact in conditions of hydrodynamic friction. Mixed problems of mechanics of a deformable body. Second All-

Union Scientific Conference: Tez. dokl. Dnepropetrov. gos. un-t, 1981.
10. VAN DYKE M.D., Perturbation Methods in Fluid Mechanics. Academic Press, N.Y., 1964.

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AN ASYMPTOTIC APPROACH TO THE PROBLEMS OF THE THEORY OF ELASTICITY OF BODIES OF FINITE DIMENSIONS*

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A method is developed to solve the equilibrium problems of elastic bodies of fixed dimensions, based on a separation of the boundary-layer part of the solution by considering the problem for a half-strip. A closed solution in quadratures is constructed for the half-strip with a free lateral face and with given normal displaced longitudinal boundaries, using both symmetric and antisymmetric loading. When the normal stresses on the front boundaries are specified, the problem reduces to an integral equation of the first kind in a semi-infinite interval, the inversion of which is obtained by reduction to an infinite system of algebraic equations. The approach considered for problems of bodies of finite dimensions is asymptotic with respect to the small parameter characterizing the body's thickness. Testing of the method on a plane problem for an elastic rectangle enables the range of variation of this parameter to be investigated, in which this procedure is fairly accurate. In the example considered, for the case of a rectangular area, the stresses are found and compared with the results obtained earlier by other methods. The nature of the influence of the boundary layer on the stress distribution inside the body is investiaged.

Asymptotic methods, used for bodies of slab configuration, one of whose characteristic dimensions (thickness) is significantly less than the other two /1-4/, can obviously be classified into three types. The first of them /5,6/ is characterized by the application of joined asymptotic expansions to a certain class of solutions of the equations of the theory of elasticity, namely uniform solutions. The second type /7,8/ is distinguished by an asymptotic analysis of the equations of the theory of elasticity. From this it is clear that, to separate the boundary-layer part of the solution, it is enough, to a first approximation, to examine the two-dimensional problem and the problem of torsion for the half-strip, the lateral face of which is combined with the generating lateral surface of the plate at a given point. Finally, the third class includes the Vekua-Poniatovskii theory /9,10/, in which asymptotic methods are also developed /11/. In this sense this paper relates to the second of these methods.

1. In a Cartesian system of coordinates x, y we will examine the statical problem of the two-dimensional deformation of an elastic isotropic half-strip (the x-axis is the axis of symmetry and is directed parallel to the side faces, and the y-axis lies in the place of the half-strip end face) with the following boundary conditions:

$$x = 0,\ \tau_{xy} = 0,\ \sigma_x = 0 \qquad (1.1)$$
$$y = \pm 1,\ \tau_{xy} = 0,\ u_y = \pm f(x)$$

We will assume the boundary function $f(x)$ to be fairly smooth.

In this symmetric case, (the extension-compression case), the problem was examined in /12/, where its closed solution was obtained, based on the theory of dislocations. Here a similar result will be obtained using the well-known classical representation of the solution for a half-strip (ν is Poisson's ratio) /2/

$$u_x = -\frac{2}{\pi}\int_0^\infty \Big[A(s)\operatorname{ch} sy + C(s)\Big(\frac{3m-4}{ms}\operatorname{ch} sy + y\operatorname{sh} sy\Big)\Big]\sin sx\, ds +$$
$$\sum_{n=1}^\infty (B_n + D_n x)e^{-\alpha_n x}\cos\alpha_n y$$

$$u_y = \frac{2}{\pi}\int_0^\infty [A(s)\operatorname{sh} sy + C(s)\, y\operatorname{ch} sy]\cos sx\, ds +$$

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